

Topological Reverberations in Flat Space-times

G.I. Gomero*, M.J. Rebouças[§], A.F.F. Teixeira[†], and A. Bernui[‡]

Centro Brasileiro de Pesquisas Físicas

Departamento de Relatividade e Partículas

Rua Dr. Xavier Sigaud 150

22290-180 Rio de Janeiro – RJ, Brazil

(February 7, 2008)

We study the role played by multiply-connectedness in the time evolution of the energy $E(t)$ of a radiating system that lies in static flat space-time manifolds \mathcal{M}_4 whose $t = \text{const}$ spacelike sections \mathcal{M}_3 are compact in at least one spatial direction. The radiation reaction equation of the radiating source is derived for the case where \mathcal{M}_3 has any non-trivial flat topology, and an exact solution is obtained. We show that the behavior of the radiating energy $E(t)$ changes remarkably from exponential damping, when the system lies in \mathcal{R}^3 , to a reverberation pattern (with discontinuities in the derivative $\dot{E}(t)$ and a set of relative minima and maxima) followed by a growth of $E(t)$, when \mathcal{M}_3 is endowed with any one of the 17 multiply-connected flat topologies. It emerges from this result that the compactness in at least one spatial direction of Minkowski space-time is sufficient to induce this type of topological reverberation, making clear that topological fragilities can arise not only in the usual cosmological modelling, but also in ordinary flat space-time manifolds. An explicit solution of the radiation reaction equation for the case where $\mathcal{M}_3 = \mathcal{R}^2 \times \mathcal{S}^1$ is discussed in details, and graphs which reveal how the energy varies with the time are presented and analyzed.

I. INTRODUCTION

In general relativity, as well as in any metrical theory of gravitation of some generality and scope, a common approach to cosmological modelling commences with a four-dimensional space-time manifold (representing the physical events) endowed with a Lorentzian metric (necessary to ensure the local validity of the well established special relativity theory). Representing the physical phenomena in this manifold we have fields satisfying appropriate local differential equations (the physical laws). Finally, fields and geometry are coupled according to the corresponding gravitational theory one is dealing with. The space-time geometries arising as solutions of the gravitational field equations constrain to some extent the dynamical behavior of the physical fields. This eminently metrical approach to model the physical world has led a number of physicists to implicitly (or explicitly) restrict their studies to purely geometric features of space-time, either by ignoring the role of topology or by considering just a limited set of topological alternatives for the space-time manifold.

If, on the one hand, the topological properties of a manifold antecede and are more fundamental than the metrical features and the differentiable structure on which tensor analysis is based upon, on the other hand, it is well known that geometry constrains but it does not dictate the topology of a space-time. It is therefore important to determine whether and (or) to what extent physical results concerning a space-time geometry depend on or are somehow constrained by the topology of the underlying manifold.

In a recent work [1] we studied the role played by topology in the time evolution of the energy $E(t)$ of a radiating system in static flat FRW space-time manifolds \mathcal{M}_4 whose $t = \text{const}$ spacelike sections \mathcal{M}_3 are endowed with different topologies, namely the simply-connected Euclidean manifold \mathcal{R}^3 , and six topologically non-equivalent flat multiply-connected orientable *compact* 3-manifolds [2] – [5]. Clearly these flat space-time manifolds \mathcal{M}_4 are orientable and decomposable into $\mathcal{M}_4 = \mathcal{R} \times \mathcal{M}_3$. The radiating system we have examined, which has been used to study a wide class of oscillating phenomena [6] – [16], is formed by a pointlike harmonic oscillator (energy source) coupled with a massless scalar field (scalar radiation waves propagating at speed of light). Through a *numerical* integration of the time evolution equation for the harmonic oscillator (with the topological constraints suitably considered), we have shown that there is an exponential damping of the energy $E(t)$ of the harmonic oscillator when the spacelike

section \mathcal{M}_3 is the Euclidean space \mathcal{R}^3 , whereas when \mathcal{M}_3 is endowed with one of the six compact and orientable flat topologies the energy $E(t)$ exhibits a loosely speaking reverberation pattern, with discontinuities in the derivative of $E(t)$ and relative minima and maxima (both the discontinuities and the extrema are due to the travelling waves which “reflect” from the “topological walls”), followed by a growth of the energy with the time. We have also shown that, for these six compact cases, the energy $E(t)$ diverges exponentially when $t \rightarrow \infty$, in striking contrast with the damping of $E(t)$ in the \mathcal{R}^3 case. This unexpected divergent behavior of the energy for flat space-times with compact spacelike sections illustrates that (*totally*) compact topologies may give rise to rather important dynamic changes in the behavior of a physical system. This type of sensitivity has been referred to as topological fragility and can occur without violation of any local physical law [17].

In the present work we extend these investigations by performing a rigorous *non-numerical* study of the behavior of the same physical system under a less restrictive topological setting, namely we shall consider static flat space-time manifolds \mathcal{M}_4 whose $t = \text{const}$ spacelike sections \mathcal{M}_3 are compact in at least one spatial direction. Moreover, no assumption will be made as to whether or not these 3-manifolds are orientable. This amounts to saying that the results of the present work hold for all flat space-time manifolds whose $t = \text{const}$ spacelike sections are endowed with one of the 17 flat topologies discussed by Wolf [2], Ellis [3], and others [18] (see also Gomero [5]).

It should be stressed from the outset that in spite of being the same physical system discussed in [1], the present work generalizes the results of that article in several respects. Firstly, the underlying topological setting is now much more general in that here we do not assume that the spacelike 3-manifolds \mathcal{M}_3 are compact and orientable. Secondly, contrarily to the *numerical* integration performed in [1], in the present paper we have obtained a closed *exact* solution of the evolution equation for the harmonic oscillator in the above-mentioned general topological setting. Thirdly, the case study we have examined in section IV, where we obtain the explicit solution for the radiation reaction equation and discuss in details the graphs which reveal the time evolutions of $E(t)$ and $\dot{E}(t)$, is completely new. Finally here we have also performed a lengthy discussion on the conservation and on the balance of the total energy of our physical system, making apparent how (here as well as in ref. [1]) the total energy of the whole system is conserved for any time $t > 0$.

The plan of this article is as follows. We derive (section II) the time evolution equation for the harmonic oscillator (also referred to as radiation reaction equation) when \mathcal{M}_3 is endowed with a non-trivial flat topology, i.e., when the $t = \text{const}$ spacelike sections are any multiply-connected (compact in at least one direction) flat manifold \mathcal{M}_3 . It turns out that the evolution equation for this case is formally the same as that obtained by Bernui *et al.* [1] for the case where \mathcal{M}_3 is any (totally) compact and orientable 3-manifold. However, they differ in the non-homogeneous term that accounts for the difference in the topology — distinct degrees of connectedness of the spacelike sections give rise to different non-homogeneous terms. We also emphasize in section II that since our system is composed only of test fields in static flat space-time backgrounds, there is no need to consider the back-reaction of the fields on the geometry.

We present in section III an *exact* solution of the evolution equation for the harmonic oscillator in flat space-time whose $t = \text{const}$ spacelike sections are any multiply-connected flat 3-manifold. Through a heuristic analytical reasoning we also show in section III that this exact solution $Q(t)$ and the corresponding energy of the harmonic oscillator $E(t)$ both exhibit a divergent *asymptotical* exponential behavior. Clearly the energy can be limitlessly “extracted” from the interaction term in the multiply-connected cases only because the system permits this extraction, since it is not bounded from below. But as we shall discuss in this work it is indeed the topology that “excites” this available “physical mode” of our system. The balance and conservation of the total energy of our system is also discussed in section III, where we show that it is finite and conserved for all finite time $t > 0$.

Explicit exact solution of the evolution equation for the case where $\mathcal{M}_3 = \mathcal{R}^2 \times \mathcal{S}^1$ is discussed in details in section IV. For this special case we also present and analyze graphs which reveal how the amplitude $Q(t)$ and the energy $E(t)$ of the *overdamped* harmonic oscillator vary with the time. The graph of $E(t)$ presents a reverberation pattern with relative minima and maxima, and discontinuities of the derivative $\dot{E}(t)$. The topological origin of the reverberations for the overdamped oscillator is also made clear and stressed in that section. The graphs in this section also confirm (within the limit of accuracy of the plots, of course) the divergent asymptotical exponential behavior

for both $Q(t)$ and $E(t)$, which is shown to take place in section III. These results make apparent that, contrarily to what one might infer from [1], it is not necessary to have flat space-time manifolds with (totally) compact spacelike sections \mathcal{M}_3 for the topological induction of the reverberations and (or) the divergent behavior of the energy $E(t)$ — the compactness of \mathcal{M}_3 in just one direction suffices.

We begin the section V by emphasizing the importance of topological considerations in physical problems, and give an example that makes explicit how the overall electric charge of universes whose $t = \text{const}$ spacelike sections are orientable and compact flat 3-manifolds is related to the topology. We also show that topological fragility can arise not only in the usual cosmological modelling, but also in the ordinary multiply-connected flat space-time manifolds we consider in the present paper. There we stress the origin of the topological reverberations we found, and show that it constitutes an example of topological fragility, which takes place without violation of any local physical law. The extent to which this paper generalizes ref. [1] is discussed in details in section V. Finally the relation of our results with cosmology is also addressed.

II. RADIATION REACTION EQUATION

The flat space-time manifolds we shall be concerned with are decomposable into $\mathcal{M}_4 = \mathcal{R} \times \mathcal{M}_3$, where the real line \mathcal{R} represents a well defined global time of Minkowski space-time. Corresponding to the possible topologically distinct 3-manifolds \mathcal{M}_3 there exists a simply-connected covering manifold \mathcal{R}^3 , such that each manifold \mathcal{M}_3 is obtained from \mathcal{R}^3 by identifying points which are equivalent under the action of a discrete subgroup of isometries of the Euclidean space \mathcal{R}^3 without fixed points. In other words, each manifold \mathcal{M}_3 is obtained by forming the quotient space $\mathcal{M}_3 = \mathcal{R}^3/\Gamma$, where Γ is a discrete group of isometries of \mathcal{R}^3 acting freely on \mathcal{R}^3 and referred to as the covering group of \mathcal{M}_3 (for a complete classification of Euclidean 3-manifolds see Wolf [2]).

The physical system we shall be concerned throughout this paper is represented by a pointlike harmonic oscillator coupled with a scalar field. This system has been treated by many people, since the work by Schwalb and Thirring [6] in 1964, as a simplified model to study a few features of oscillating (radiating) phenomena [6] – [16]. When the underlying manifold is \mathcal{R}^3 this system may serve as a model for an electric dipole coupled to electromagnetic radiation or an impurity atom interacting with acoustical waves [6]. It has also been used as simple model to parallel classical electrodynamics in the dipole approximation as discussed by Kampen [19] and Kramers [20]. It is worth mentioning that similar systems have also been used to mimic the basic properties, and thus to study the most relevant features of pulsating stellar systems [21] – [23]. The essential idea that permeates these latter papers is to work with simple models of oscillating fields interacting with oscillating sources in order to develop some analytical physical background to understand the relationship between gravitational waves and their sources (radiation emission and radiation reaction).

Consider that the harmonic oscillator of our physical system is located at an arbitrary point $q \in \mathcal{M}_3$. Without loss of generality, one can assume that the covering map $\pi : \mathcal{R}^3 \rightarrow \mathcal{M}_3$ maps the origin to q , namely $\pi(0, 0, 0) = q$. In what follows we shall denote by \mathcal{O}_q the orbit $\pi^{-1}(q)$ of $(0, 0, 0)$ under the action of Γ on \mathcal{R}^3 (see fig. 1).

Now, the time evolution of the harmonic oscillator at $q \in \mathcal{M}_3$ can be obtained by studying the equivalent system on the covering manifold, which is formed by an infinite set of indistinguishable harmonic oscillators each one located at a point of the orbit \mathcal{O}_q , subject to identical sets of initial conditions and interacting with the scalar field φ in the same way as that of the original oscillator at $q \in \mathcal{M}_3$ (see fig. 1). Thus, the dynamics of our physical system in \mathcal{M}_3 can be derived from the following functional action for the equivalent system in the universal covering space \mathcal{R}^3 :

$$S = S_f + S_o + S_i, \quad (2.1)$$

where S_f is a scalar field term, S_o corresponds to the oscillators, and S_i is a coupling term between the scalar field and each harmonic oscillator. These three terms are, respectively, given by

$$S_f = \frac{1}{2} \int d^4x \, \eta^{\mu\nu} \partial_\mu \varphi(t, \vec{x}) \partial_\nu \varphi(t, \vec{x}), \quad (2.2)$$

$$S_o = \frac{1}{2} \sum_{\vec{p} \in \mathcal{O}_q} \int dt [\dot{Q}_{\vec{p}}^2(t) - \omega^2 Q_{\vec{p}}^2(t)] , \quad (2.3)$$

$$S_i = \lambda \sum_{\vec{p} \in \mathcal{O}_q} \int d^4x \rho_{\vec{p}}(\vec{x}) \theta(t) \varphi(t, \vec{x}) Q_{\vec{p}}(t) , \quad (2.4)$$

where $t \in (0, \infty)$, $\varphi(t, \vec{x}) = \varphi(t, \gamma \vec{x})$ is the massless scalar field, $Q_{\vec{p}}(t)$ is the amplitude of the oscillator located at $\vec{p} \in \mathcal{O}_q$, overdot represents derivative with respect to the time t , ω is the angular frequency of each oscillator and $\lambda \neq 0$ is the coupling constant. Hereafter, ρ is a non-negative normalized density function ($\int \rho = 1$) with compact support (corresponding to the region of interaction) contained in the interior of a fundamental domain of \mathcal{M}_3 and centered at the origin, and finally $\rho_{\vec{p}}(\vec{x}) = \rho(\gamma^{-1}\vec{x})$, where $\gamma \in \Gamma$ and $\gamma(0, 0, 0) = \vec{p}$ [note that $\rho_{\vec{0}}(\vec{x}) = \rho(\vec{x})$]. The step function $\theta(t)$ is used to indicate that the interaction starts at $t = 0$. It should be noticed that the interaction term (2.4) clearly is not bounded from below, potentially permitting a limitless extraction of energy by the oscillator. However, as we shall discuss later in this article, it is the topology of the 3-space that “excites” this physically available mode of the system.

A word of clarification is in order here. In dealing with the behavior of physical fields in the curved space-times of general relativity one has to consider that these fields are not only influenced by, but they also have gravitational effects, i.e., they change the geometry. In most ordinary situations, however, these back-reaction effects are small enough that can be neglected. In such cases the fields are treated as *test* fields in the corresponding curved space-time background. Had we considered our fields $Q(t)$ and $\varphi(t, \vec{x})$ as *non-test* fields in a *non-static curved* space-time we would have to take into account the back-reaction of the fields on the geometry, and so besides S_i a new interaction term would have to be added to the action (2.1). However, our backgrounds are *static flat* space-times (endowed with any one of the 17 possible non-trivial flat topologies), and our system contains only *test* fields with a mutual interaction, thus there is no place for such a term in the action of our system.

Varying the action (2.1) with respect to φ and Q one obtains the coupled equations of motion of the equivalent system, namely

$$\square \varphi(t, \vec{x}) = \lambda \sum_{\vec{p} \in \mathcal{O}_q} \rho_{\vec{p}}(\vec{x}) \theta(t) Q_{\vec{p}}(t) , \quad (2.5)$$

$$\ddot{Q}_{\vec{p}}(t) + \omega^2 Q_{\vec{p}}(t) = \lambda \int d^3x \rho_{\vec{p}}(\vec{x}) \theta(t) \varphi(t, \vec{x}) , \quad (2.6)$$

where \square denotes the d’Alembertian operator on \mathcal{R}^3 . Note that the sum in (2.5) is not an infinite sum. Actually owing to the disjoint compact supports of the functions $\rho_{\vec{p}}(\vec{x})$ the right-hand side of that equation contains at most one summand which does not vanish.

To obtain the radiation reaction equation from the above equations (2.5) and (2.6) we recall that the general solution of eq. (2.5), considered as an initial value problem, can be written in the form $\varphi(t, \vec{x}) = \varphi_I(t, \vec{x}) + \varphi_H(t, \vec{x})$, where $\varphi_H(t, \vec{x})$ satisfies the corresponding homogeneous equation, and where $\varphi_I(t, \vec{x})$ indicates the solution of the inhomogeneous equation. From now on we shall assume, for simplicity, that the whole energy of the system is initially stored in the oscillator. Accordingly, we shall use as initial conditions for the equations (2.5) and (2.6) that $\varphi(0, \vec{x}) = \dot{\varphi}(0, \vec{x}) = 0$ and that $Q(0) = \alpha$, $\dot{Q}(0) = \beta$, where α and β are real arbitrary constants. These conditions imply that $\varphi_H(t, \vec{x}) = 0$, and therefore $\varphi(t, \vec{x}) = \varphi_I(t, \vec{x})$.

Now since the interaction begins at time $t = 0$, the Green function for the d’Alembertian operator in (2.5) is clearly given by¹

¹Clearly there are two equivalent approaches to build the Green function of the wave operator, which depend on whether it is considered on the quotient manifold $\mathcal{M}_3 = \mathcal{R}^3/\Gamma$, or on the covering space \mathcal{R}^3 . On \mathcal{M}_3 , as it has been considered in [1], it has to be invariant under the covering transformations. On \mathcal{R}^3 , however, which we are considering in this section, the

$$G(t, \vec{x}; \tau, \vec{y}) = \frac{\delta(t - \tau - |\vec{x} - \vec{y}|)}{4\pi |\vec{x} - \vec{y}|}. \quad (2.7)$$

Hence, the solution for equation (2.5) can be formally written as

$$\varphi(t, \vec{x}) = \frac{\lambda}{4\pi} \sum_{\vec{r} \in \mathcal{O}_q} \int d^3y \frac{\rho_{\vec{r}}(\vec{y})}{|\vec{x} - \vec{y}|} \theta(t - |\vec{x} - \vec{y}|) Q_{\vec{r}}(t - |\vec{x} - \vec{y}|), \quad (2.8)$$

where we have used \vec{r} as dummy index instead of \vec{p} .

Finally, inserting (2.8) into equation (2.6) one easily obtains the radiation reaction equations for the oscillators, namely

$$\ddot{Q}_{\vec{p}}(t) + \omega^2 Q_{\vec{p}}(t) = \frac{\lambda^2}{4\pi} \sum_{\vec{r} \in \mathcal{O}_q} \int d^3x d^3y \frac{\rho_{\vec{p}}(\vec{x}) \rho_{\vec{r}}(\vec{y})}{|\vec{x} - \vec{y}|} \theta(t - |\vec{x} - \vec{y}|) Q_{\vec{r}}(t - |\vec{x} - \vec{y}|). \quad (2.9)$$

As the infinite set of identical harmonic oscillators (each one located at a point of \mathcal{O}_q) are subject to the same set of physical constraints, the amplitudes $Q_{\vec{p}}$ must evolve identically. Thus, the set of radiation reaction equations (2.9) reduces to just one differential equation, namely

$$\ddot{Q}(t) + \omega^2 Q(t) = \frac{\lambda^2}{4\pi} \sum_{\vec{r} \in \mathcal{O}_q} \int d^3x d^3y \frac{\rho(\vec{x}) \rho_{\vec{r}}(\vec{y})}{|\vec{x} - \vec{y}|} \theta(t - |\vec{x} - \vec{y}|) Q(t - |\vec{x} - \vec{y}|), \quad (2.10)$$

which holds for each oscillator in $\mathcal{O}_q \subset \mathcal{R}^3$, and whose solution $Q(t)$ gives the time evolution for the amplitude of the harmonic oscillator in \mathcal{M}_3 .

One might think at first sight that the sums on the right hand side of eqs. (2.8) – (2.10) are divergent, and a regularisation scheme is needed to cope with the divergences. However, this is not true because for any finite time $t > 0$ the sums in those equations are in fact finite owing to the disjoint compact supports of the functions $\rho_{\vec{p}}(\vec{x})$ together with the cut-off effects of the step function $\theta(t)$. In other words, we do not have formal series to be carefully handled, but finite expressions.

To deal with the pointlike coupling between the harmonic oscillator and the scalar field $[\rho(\vec{x}) \rightarrow \delta^3(\vec{x})]$, we shall consider an infinite family of oscillator–field systems (n -system, for short), with the same coupling constant $\lambda \neq 0$, and such that each element of the n -system is characterized by a *bare* angular frequency ω_n and density function ρ_n , satisfying the generalized Aichelburg-Beig condition [8], namely

$$\omega_n^2 - \frac{\lambda^2}{4\pi} \int d^3x d^3y \frac{\rho_n(\vec{x}) \rho_n(\vec{y})}{|\vec{x} - \vec{y}|} \equiv \Omega^2 > 0, \quad (2.11)$$

where Ω is the same constant for each member of the n -system, and $\{\rho_n\}$ is a δ -sequence in the sense of distributions [24]. It should be noticed that the inequality (2.11) is a sufficient condition for having radiation damping of the energy $E(t)$ in the pointlike coupling limit case when the oscillator-field system lies in the Minkowski space-time manifold $\mathcal{M}_4 = \mathcal{R} \times \mathcal{R}^3$ (see, for example, ref. [8]).

Now, since eq. (2.10) holds for each member of the n -system, using relation (2.11) one finds

$$\begin{aligned} \ddot{Q}_n(t) + 2\Gamma \int d^3x d^3y \rho_n(\vec{x}) \rho_n(\vec{y}) \frac{Q_n(t) - \theta(\tau) Q_n(\tau)}{|\vec{x} - \vec{y}|} \\ + \Omega^2 Q_n(t) = 2\Gamma \sum_{\vec{r} \in \tilde{\mathcal{O}}_q} \int d^3x d^3y \frac{\rho_n(\vec{x}) \rho_{n,\vec{r}}(\vec{y})}{|\vec{x} - \vec{y}|} \theta(\tau) Q_n(\tau), \end{aligned} \quad (2.12)$$

invariance under Γ must not be imposed. Note additionally that the covering transformations have been taken into account in eqs. (2.3) – (2.6) and in the subsequent equations (2.9) – (2.13).

where we have set $\tau = t - |\vec{x} - \vec{y}|$, $2\Gamma = \lambda^2/4\pi$ and $\tilde{\mathcal{O}}_q = \mathcal{O}_q - \{(0, 0, 0)\}$.

In the limit $n \rightarrow \infty$ (see Appendix) we have that the following equation holds for all $t > 0$:

$$\ddot{Q}(t) + 2\Gamma \dot{Q}(t) + \Omega^2 Q(t) = \sum_{k=1}^{\infty} C_k \theta(t - t_k) Q(t - t_k), \quad (2.13)$$

where $C_k = 2\Gamma D_k/t_k$, D_k is the number of points $\vec{r} \in \tilde{\mathcal{O}}_q$ such that $|\vec{r}| = t_k$, the sequence $\{t_k\}$ is ordered by increasing values, and clearly $t_k \rightarrow \infty$ when $k \rightarrow \infty$.

It should be noticed that the right-hand side of the radiation reaction equation (2.13) formally contains an infinite countable number of retarded terms of topological origin. However, due to the θ function, at any given finite time t only a finite number of terms will contribute to the sum of (2.13). These terms are different for distinct multiply-connectedness of the 3-manifolds \mathcal{M}_3 , and vanish for the simply-connected manifold \mathcal{R}^3 . Loosely speaking they give rise to “reflected” waves from the “topological walls”, and thus account for both the extrema of $E(t)$ and the discontinuities of $\dot{E}(t)$. In other words they are responsible for the topological reverberations. Clearly for the simply-connected manifold \mathcal{R}^3 , there is an exponential decay of $E(t)$ and no reverberation takes place in this case.

III. SOLUTION AND ASYMPTOTICAL BEHAVIOR

In this section we shall present an exact recursive solution of the radiation reaction equation (2.13), which holds for any multiply-connected flat manifold \mathcal{M}_3 . To this end, we first consider the differential operator

$$D = d^2 + 2\Gamma d + \Omega^2, \quad (3.1)$$

which acts on real-valued piecewise smooth functions $f(t)$ according to

$$Df = \ddot{f} + 2\Gamma \dot{f} + \Omega^2 f. \quad (3.2)$$

It is straightforward to show that if $g(t)$ is some integrable function, then the solution of the equation

$$Df = g \quad (3.3)$$

with initial conditions $f(0) = \dot{f}(0) = 0$ is

$$f(t) = e^{-\Gamma t} \int_0^t e^{\Gamma \tau} h(t - \tau) g(\tau) d\tau, \quad (3.4)$$

with $\mu^2 = \Omega^2 - \Gamma^2$ and

$$h(t) = \begin{cases} \mu^{-1} \sin \mu t & \text{if } \mu^2 > 0, \\ t & \text{if } \mu^2 = 0, \\ \nu^{-1} \sinh \nu t & \text{if } \mu^2 = -\nu^2 < 0. \end{cases} \quad (3.5)$$

In what follows we shall discuss in detail how to find the solution of (2.13) for the underdamped case, in which $\mu^2 > 0$. The solution for the remaining two cases, however, can be similarly worked out, and for the sake of brevity we will present later only the final results.

Our purpose now is to find a real-valued function $Q : [0, \infty) \rightarrow \mathcal{R}$ of class C^1 such that $Q(0) = \alpha$ and $\dot{Q}(0) = \beta$ and satisfying (2.13), which in terms of the operator D can be rewritten as

$$DQ(t) = \sum_{k=1}^{\infty} C_k \theta(t - t_k) Q(t - t_k), \quad (3.6)$$

where the C_k are positive real numbers and $\{t_k\}$ is a strictly increasing sequence of positive real numbers ($0 < t_1 < t_2 < \dots < t_k$), such that $t_k \rightarrow \infty$ when $k \rightarrow \infty$.

Thus, for $f(t) = Q(t) - s_0(t)$ and taking into account (3.3) and (3.4) the solution of (3.6) can be written as

$$Q(t) = s_0(t) + \frac{1}{\mu} \sum_{k=1}^{\infty} C_k \theta(t - t_k) e^{-\Gamma(t-t_k)} \int_0^{t-t_k} e^{\Gamma\tau} \sin \mu(t - t_k - \tau) Q(\tau) d\tau, \quad (3.7)$$

where the function s_0 must satisfy the homogeneous equation $D s_0(t) = 0$ with the conditions $s_0(0) = \alpha$, $\dot{s}_0(0) = \beta$. Clearly one can rewrite the solution (3.7) in the form

$$Q(t) = s_0(t) + \sum_{k=1}^{\infty} C_k \theta(t - t_k) Q_1(t - t_k), \quad (3.8)$$

where, from (3.3) and (3.4), the function

$$Q_1(t) = \frac{1}{\mu} e^{-\Gamma t} \int_0^t e^{\Gamma\tau} \sin \mu(t - \tau) Q(\tau) d\tau \quad (3.9)$$

fulfils the differential equation $D Q_1(t) = Q(t)$ with the initial conditions $Q_1(0) = \dot{Q}_1(0) = 0$.

The solution (3.8) is given in an implicit form. An explicit form can be obtained, though. Indeed, as we are concerned only with real-valued functions defined for $t \in [0, \infty)$, if we recursively define functions s_n by

$$D s_n(t) = s_{n-1}(t) \quad \text{for } n \geq 1, \quad (3.10)$$

$$D s_0(t) = 0, \quad (3.11)$$

with the conditions $s_n(0) = \dot{s}_n(0) = 0$ for $n \geq 1$, and functions $Q_n(t)$ by

$$D Q_n(t) = Q_{n-1}(t) \quad \text{for } n \geq 1, \quad (3.12)$$

$$Q_0(t) = Q(t), \quad (3.13)$$

with $Q_n(0) = \dot{Q}_n(0) = 0$ for $n \geq 1$, then we can show by induction that

$$Q_n(t) = s_n(t) + \sum_{k=1}^{\infty} C_k \theta(t - t_k) Q_{n+1}(t - t_k). \quad (3.14)$$

Notice that the first of these equations (for $n = 0$) is nothing but equation (3.8).

An explicit form for the solution of (3.6) can now be obtained as follows. Equations (3.10) and (3.11) can be integrated to give

$$s_0(t) = e^{-\Gamma t} (A \sin \mu t + B \cos \mu t), \quad (3.15)$$

$$s_n(t) = \frac{1}{\mu} e^{-\Gamma t} \int_0^t e^{\Gamma\tau} \sin \mu(t - \tau) s_{n-1}(\tau) d\tau \quad \text{for } n \geq 1, \quad (3.16)$$

where $A = (\beta + \alpha \Gamma) / \mu$, $B = \alpha$, and where obviously we have used (3.2) – (3.5). On the other hand, if we define

$$\begin{aligned} q_0(t) &= s_0(t), \\ q_1(t) &= \sum_{k_1} C_{k_1} \theta(t - t_{k_1}) s_1(t - t_{k_1}), \\ q_2(t) &= \sum_{k_1, k_2} C_{k_1} C_{k_2} \theta(t - t_{k_1} - t_{k_2}) s_2(t - t_{k_1} - t_{k_2}), \\ &\vdots \\ q_i(t) &= \sum_{k_1, k_2, \dots, k_i} \left(\prod_{j=1}^i C_{k_j} \right) \theta(t - \sum_{j=1}^i t_{k_j}) s_i(t - \sum_{j=1}^i t_{k_j}), \end{aligned} \quad (3.17)$$

then the explicit form of the solution of equation (3.6) is simply given by

$$Q(t) = \sum_{i=0}^{\infty} q_i(t) . \quad (3.18)$$

It should be noticed that since the above solution is ultimately given in terms of s_0 , equations (3.15) – (3.18) make apparent that $Q(t)$ is completely determined by the pair of initial conditions $[Q(0) = \alpha, \dot{Q}(0) = \beta]$ as one would have expected from the outset.

Another important point to be stressed regarding the above solution is that the right-hand side of (3.18) is not an infinite sum as it appears at first sight. Actually, for a given finite time t the above sum will clearly be truncated by the presence of the step function $[\theta(t) = 0 \text{ for } t \leq 0, \theta(t) = 1 \text{ for } t > 0]$, i.e. only those $q_i(t)$ with $i < t/t_1$ will contribute to the sum (3.18). Moreover, each $q_i(t)$ as given by (3.17) is itself a finite sum, again due to the cut-off effects of the step function $\theta(t)$.

Before proceeding to the discussion of the asymptotical behavior of $Q(t)$ we mention that the solution of (2.13) for the critically damped ($\mu^2 = 0$) and overdamped ($\mu^2 = -\nu^2 < 0$) cases is again given by (3.18) with (3.17), but now with the functions s_0 and s_n given by

$$s_0(t) = e^{-\Gamma t} (A + B t) \quad \text{for} \quad \mu^2 = 0 , \quad (3.19)$$

$$s_0(t) = e^{-\Gamma t} (A \sinh \nu t + B \cosh \nu t) \quad \text{for} \quad \mu^2 = -\nu^2 < 0 , \quad (3.20)$$

$$s_n(t) = e^{-\Gamma t} \int_0^t e^{\Gamma \tau} h(t - \tau) s_{n-1}(\tau) d\tau \quad \text{for} \quad n \geq 1 , \quad (3.21)$$

where A and B are real constants, which are determined by the initial conditions, and the functions $h(t)$ are given by equations (3.5).

Now, for any non-trivial flat topology the asymptotic behavior of the solution $Q(t)$ of equation (3.6) can be figured out by a procedure similar to that used in [1]. Indeed, if one *ad hoc* assumes an exponential asymptotical behavior for Q of the form $Q(t) = \gamma \exp(\sigma t)$ with σ and γ real constants, then in the limit $t \rightarrow \infty$ equation (3.6) reduces to

$$\sigma^2 + 2 \Gamma \sigma + \Omega^2 = \sum_{k=1}^{\infty} C_k e^{-\sigma t_k} . \quad (3.22)$$

This equation has only one real solution for σ . Indeed, the right-hand side of (3.22) is a positive monotone decreasing function $r(\sigma)$ with $\lim_{\sigma \rightarrow 0} r(\sigma) = \infty$ and $\lim_{\sigma \rightarrow \infty} r(\sigma) = 0$. Thus $r(\sigma)$ lies entirely in the first quadrant of the plane and crosses it from the top-left to the bottom-right (see fig. 2). Now, since $\Gamma > 0$ then for a given pair (Γ, Ω) the left-hand side of (3.22) is a parabola curved upwards with vertex at $\sigma = -\Gamma$. Therefore, it always intersects the curve for $r(\sigma)$ in just one point, which is in the first quadrant (see fig. 2). This amounts to saying that there is only one real number $\sigma = b > 0$ solution to equation (3.22). Hence the amplitude $Q(t)$, and consequently the energy of the oscillator $E(t) = \frac{1}{2} [\dot{Q}^2(t) + \Omega^2 Q^2(t)]$, both exhibit an asymptotical exponentially divergent behavior (confirmed, within the limits of accuracy of the plots, by figures 3 and 4 for the special case of an overdamped oscillator discussed below). One can sum up by stating that the compactification in just one direction is sufficient to topologically induce a asymptotic divergent behavior of $E(t)$.

A question which naturally arises here is whether the total energy of our system is conserved. As our system contains only mutually interacting test fields $Q(t)$ and $\varphi(t, \vec{x})$ in static flat background, the energy of the overall system must be conserved. Indeed, for $t > 0$ we have $\theta(t) = 1$ and the action (2.1) – (2.4) does not depend explicitly on the time. Thus, for any well-behaved density function $\rho(\vec{x})$ the total energy of our system $\mathcal{E}(t)$ clearly is conserved, which squares with the fact that our system is a test system in a static background. For the sake of completeness, in what follows we shall show in more details how the total energy of our system $\mathcal{E}(t)$ is finite and conserved for all time $t \in (0, \infty)$ even in the pointlike limit case $\rho(\vec{x}) \rightarrow \delta^3(\vec{x})$. We shall also qualitatively discuss how the balance of $\mathcal{E}(t)$ can be handled.

From the functional action (2.1) – (2.4) the energy $\mathcal{E}(t)$ of our system can clearly be written in the form

$$\mathcal{E}(t) = \mathcal{E}_f(t) + \mathcal{E}_o(t) + \mathcal{E}_i(t) , \quad (3.23)$$

where

$$\mathcal{E}_f(t) = \frac{1}{2} \int_{\mathcal{D}} d^3x \{ \dot{\varphi}^2(t, \vec{x}) + [\nabla \varphi(t, \vec{x})]^2 \} , \quad (3.24)$$

$$\mathcal{E}_o(t) = \frac{1}{2} [\dot{Q}^2(t) + \omega^2 Q^2(t)] , \quad (3.25)$$

$$\mathcal{E}_i(t) = -\lambda Q(t) \int_{\mathcal{D}} d^3x \rho(\vec{x}) \varphi(t, \vec{x}) , \quad (3.26)$$

for $t \in (0, \infty)$. Here \mathcal{D} denotes a fundamental domain of \mathcal{M}_3 . As we have mentioned above for any time $t > 0$ the energy $\mathcal{E}(t)$ is conserved since the corresponding Hamiltonian does not depend explicitly on the time.

From equation (2.8) the interaction component $\mathcal{E}_i(t)$, for which the field $\varphi(t, \vec{x})$ is evaluated at $\vec{x} = 0$, i.e. in the location of the pointlike oscillator, clearly presents a divergent behavior of the form $|\vec{x}|^{-1}$ in the neighborhood of $\vec{x} = 0$ (see below for details). Now, to deal with the balance of the total energy $\mathcal{E}(t)$, we introduce the divergent quantity

$$\Lambda = \frac{\lambda^2}{4\pi} \lim_{r \rightarrow 0^+} \frac{1}{r} \quad (3.27)$$

in terms of which a finite energy $E_i(t)$ can be defined for all $0 < t < \infty$, namely

$$\mathcal{E}_i(t) = E_i(t) - \Lambda Q^2(t) . \quad (3.28)$$

Thus, for each time t in that interval, clearly one can extract from $\mathcal{E}_i(t)$ an infinite amount of negative energy to obtain a regularized finite negative energy $E_i(t)$.

The energy $\mathcal{E}_f(t)$ as given by (3.24) also diverges. In fact, from (2.8) one finds that in the neighborhood of $\vec{x} = 0$ the scalar field $\varphi(t, \vec{x})$ takes the form

$$\varphi(t, \vec{x}) = \frac{\lambda}{4\pi} \frac{Q(t)}{|\vec{x}|} + (n.d.) , \quad (3.29)$$

where $(n.d.)$ denotes the non-diverging terms for any $t \in (0, \infty)$. So, in the neighborhood of $\vec{x} = 0$ we have $|\nabla \varphi| = |(\lambda/4\pi) Q(t)/|\vec{x}|^2|$, and the integral corresponding to this term in eq. (3.24), namely $\frac{1}{2} \int_{\mathcal{D}} 4\pi r^2 dr |\nabla \varphi|^2$, has a divergent contribution of the form $\Lambda Q^2(t)/2$ when $r = |\vec{x}| \rightarrow 0$. Now since the contribution of the term $\dot{\varphi}^2$ to the integral on the right-hand side of (3.24) is finite for all $t \in (0, \infty)$, one can similarly introduce a regularized finite energy $E_f(t)$ by

$$\mathcal{E}_f(t) = E_f(t) + \frac{1}{2} \Lambda Q^2(t) , \quad (3.30)$$

which is clearly finite for all finite time $t > 0$.

The energy component corresponding to oscillator $\mathcal{E}_o(t)$ as given by (3.25) can be dealt with by using the frequency Ω given by eq. (2.11) in the limit case $\rho(\vec{x}) \rightarrow \delta^3(\vec{x})$, namely using $\omega^2 = \Omega^2 + \Lambda$. Indeed, with this renormalized frequency one can define

$$\mathcal{E}_o(t) = E_o(t) + \frac{1}{2} \Lambda Q^2(t) , \quad (3.31)$$

where the regularized energy $E_o(t) = \frac{1}{2} [\dot{Q}^2(t) + \Omega^2 Q^2(t)]$ is finite for all finite time $t > 0$. The energy of the oscillator $E_o(t)$ has also been denoted simply by $E(t)$ in many places of this paper.

Finally, we note that according to eqs. (3.28), (3.30) and (3.31), the divergent terms of the form $\Lambda Q^2(t)$ cancel out in the sum $\mathcal{E}_i(t) + \mathcal{E}_f(t) + \mathcal{E}_o(t)$. Thus, from (3.23) one obtains that the total energy of the system $\mathcal{E}(t)$ given by (3.23) reduces to

$$\mathcal{E}(t) = E_f(t) + E_o(t) + E_i(t) , \quad (3.32)$$

which is constant and finite for all time $t > 0$.

The fact that the total energy of the system \mathcal{E} is finite and conserved for all time $t > 0$ is undoubtedly the most important physical requisite our system obeys. However, it is the fact that the interaction energy is not bounded from below which makes possible and understandable the asymptotical divergent behavior of the energy of the oscillator and of the scalar field, since they both can “extract” energy from the interaction term. Remarkably this latent degree of freedom of our system is excited only when it lies in the multiply-connected manifolds, since when it is in \mathcal{R}^3 there is a damping of energy of the oscillator. In other words, the topology (multiply-connectedness) excites this available “physical mode” of our system. This remarkable feature of this system has not been perceived since the sixties, when it was first examined by Schwab and Thirring (see, for example, refs. [6] – [14]).

It is certainly desirable that quantum physical systems have the interaction energy bounded from below. However, at a classical level this is not an imperative. When one considers, for example, a system of two isolated pointlike charges with opposite signs in a flat space (vacuum medium), the smaller is the distance d between the charges the greater is the absolute value of potential energy of the system. Even if one assumes that the charges have already been renormalized, in the limit $d \rightarrow 0$ the potential energy formally diverges. This sort of divergences are formal, since the limit $d \rightarrow 0$ cannot be attained in practice. Furthermore, one ought to bear in mind that the classical theory used to infer this divergence does not hold for every small d (compared with, e.g., the Bohr radius). Actually, the distance d is assumed to be larger than the Bohr radius since otherwise the self-fields of the two interacting charges would be rather distorted, and one should expect much more involved physical effects. Similarly in our system the energy of the oscillator $E_o(t)$ is finite for all finite $t > 0$, but formally diverges in the limit $t \rightarrow \infty$. This limit case may perhaps be excluded on physical grounds. However, the most remarkable point here is that a well-behaved physical system when it lies in the ordinary simply-connected Euclidean manifold \mathcal{R}^3 , will exhibit an unexpected behaviour (reverberation pattern followed by a growth of $E(t)$) when it is in any possible flat manifold with non-trivial topology, without violating any local physical law.

IV. CASE STUDY

In this section we shall focus our attention on the time evolution of the harmonic oscillator in a flat space-time whose $t = \text{const}$ spacelike section is the simplest multiply-connected Euclidean 3-manifold, namely $\mathcal{M}_3 = \mathcal{R}^2 \times \mathcal{S}^1$. In this case, denoting by a the distance between two equivalent points in the covering manifold \mathcal{R}^3 , one can easily find that $C_k = 4\Gamma/t_k$, where $t_k = k a$ (k stands for a positive integer). Hence, from (3.17) and for $i > 0$ one obtains the solution

$$q_i(t) = \left(\frac{4\Gamma}{a}\right)^i \sum_{k_1, k_2, \dots, k_i} \left(\prod_{j=1}^i \frac{1}{k_j}\right) \theta(t - a \sum_{j=1}^i k_j) s_i(t - a \sum_{j=1}^i k_j), \quad (4.1)$$

which together with (3.18) and the first eq. (3.17) give the behavior of the amplitude $Q(t)$ and of the energy $E(t) = \frac{1}{2} [\dot{Q}^2(t) + \Omega^2 Q^2(t)]$ for the following different types of oscillator: (i) the underdamped ($\mu^2 > 0$) for which s_0 and s_n are given by (3.15) and (3.16); (ii) the critically damped ($\mu^2 = 0$) where (3.19) and (3.21) furnish s_0 and s_n ; (iii) the overdamped ($\mu^2 < 0$) for which s_0 and s_n are given by (3.20) and (3.21).

As far as the asymptotical behavior of $Q(t)$ for this special case ($\mathcal{M}_3 = \mathcal{R}^2 \times \mathcal{S}^1$) is concerned, the right-hand side series of (3.22) can easily be evaluated and equation (3.22) becomes

$$\sigma^2 + 2\Gamma\sigma + \Omega^2 = -\frac{4\Gamma}{a} \ln(1 - e^{-\sigma a}). \quad (4.2)$$

Clearly, for a specific oscillator, i.e. for a given pair (Γ, Ω) , this equation can be numerically solved for σ by using, for example, a computer algebra system such as Maple [25,26]. For an overdamped oscillator with $\Gamma = 6$ and $\Omega = 5$, which we shall use in our numeric calculations, taking $a = 1$ and using Maple one easily obtains the approximate solution $\sigma = b = 0.35$.

In the remainder of this section we shall discuss the figures 3 and 4 and make some further remarks. We begin by emphasizing that we have chosen to discuss the dynamical behavior of both the amplitude and the energy of the oscillator in a space-time whose spacelike section is $\mathcal{M}_3 = \mathcal{R}^2 \times \mathcal{S}^1$ because this is the simplest multiply-connected flat manifold (since it is compact in just one direction) having therefore the lowest degree of connectedness for a flat 3-manifold [1]. This amounts to saying that the dynamical effects exhibited by the physical quantities in this manifold will also be present in any other multiply-connected flat 3-manifold. As a matter of fact, the greater is the degree of connectedness of \mathcal{M}_3 the more reinforced will be these dynamical effects of topological nature. The limiting simply-connected case \mathcal{R}^3 will manifest no sign of such effects. A second point of general order which is worth noting is that we have decided to plot the graphs for an overdamped oscillator because we knew from the outset that the amplitude $Q(t)$ and the energy $E(t)$ for this oscillator, when in \mathcal{R}^3 , exhibit an exponential decay with no relative extrema. This choice amounts to freezing the degrees of oscillations, and therefore rules out any sort of resonance, since clearly there is no frequency to be coupled with to resonate. Besides, our choice also makes easier and neater the comparison of the dynamical behavior of the oscillator in the simply and multiply connected cases. Taking into account these considerations we have plotted the figures 3 and 4 for an overdamped harmonic oscillator with $\Gamma = 6$, $\Omega = 5$ and with initial conditions $Q(0) = 1$ and $\dot{Q}(0) = 0$, in a flat space-time whose $t = \text{const}$ spacelike section \mathcal{M}_3 is endowed with the topology $\mathcal{R}^2 \times \mathcal{S}^1$, and where we have taken the distance $a = 1$ in eqs. (4.1) and (4.2). The analysis of these figures is given in what follows.

Figure 3 shows the graph for the amplitude $Q(t)$ which exhibits relative maxima and minima followed by a growth of $Q(t)$. This time evolution of the amplitude sharply contrasts with the behavior of the amplitude $Q(t)$ for the same oscillator in \mathcal{R}^3 , where neither relative extrema nor growth of $Q(t)$ take place. This figure also contains the graph of the function $\exp(-bt)Q(t)$, which within the accuracy of numeric calculations performed for the plots confirms an exponential form for the divergent behavior suggested by the graph of $Q(t)$ and shown to indeed take place in the previous section. In this graph we have used $b = 0.35$, which is the root of eq. (4.2) for $\Gamma = 6$, $\Omega = 5$ and $a = 1$.

Figure 4 shows the behavior of the energy $E(t)$ which exhibits a few relative minima and maxima followed by an asymptotic divergent behavior. This divergent behavior is clearly in agreement with the behavior of $Q(t)$ when $t \rightarrow \infty$ shown in figure 3. The time evolution of the energy for the oscillator in this multiply-connected manifold is in striking contrast with the behavior of the energy $E(t)$ in the simply-connected case \mathcal{R}^3 , where there is an exponential damping of $E(t)$ with no relative extrema. Figure 4 also contains the graph of the derivative $\dot{E}(t)$, exhibiting discontinuities at each $t = t_k = k a$ (k stands for a positive integer), which again do not occur in the simply-connected case \mathcal{R}^3 . The graph of $\dot{E}(t)$ also reveals that $E(t)$ has a finite number of extrema.

It should be also noticed that the derivative $\dot{E} = \dot{Q}(\dot{Q} + \Omega^2 Q)$ of the energy function exhibits discontinuities not only in the specific case handled in this section, but also in the general underlying topological setting of this work. Indeed, from equation (2.13) one easily obtains

$$\dot{E}(t) = 2\Gamma \dot{Q}(t) \left[\sum_{k=1}^{\infty} \frac{D_k}{t_k} \Theta(t - t_k) Q(t - t_k) - \dot{Q}(t) \right]. \quad (4.3)$$

From this equation one has that the discontinuities occur at $t = t_k$, i.e. they come about each time a new term $Q(t - t_k)$ is taken into account in the right-hand side of eq. (2.13).

The relative extrema and asymptotic divergent behavior of $E(t)$ as well as the discontinuities of $\dot{E}(t)$, which occur only in the multiply-connected flat manifolds, are due to the retarded terms of topological origin in the evolution equation (2.13). These topological terms are different for distinct connectedness of the $t = \text{const}$ section \mathcal{M}_3 of the flat space-time manifold, and vanish for the simply-connected case \mathcal{R}^3 . They are ultimately responsible for those features (extrema, discontinuities and asymptotic behavior) of $E(t)$. In brief, as it has been made apparent for the overdamped oscillator, both the suppression of the radiation damping and the reverberation pattern of $E(t)$ in those multiply-connected flat space-times are of purely topological origin, and on the other hand, evince that our system is topologically fragile [17].

V. CONCLUDING REMARKS

Since the physical laws are usually expressed in terms of local differential equations the topological considerations may not be prominent at first sight. Nevertheless they are often necessary in many problems of physics. In general relativity and cosmology, which handles the dynamic and the global structure of space-time manifolds, the study of topological features is even more significant and acquires a dynamical meaning in a sense.

Topological considerations are necessary in many other situations in physics. When one considers, for example, the electric fields produced by bounded sources, one quite often chooses as boundary condition that the field vanishes at spatial infinity. This choice is possible and even convenient if the $t = \text{const}$ spacelike section of the Minkowski space-time \mathcal{M}_4 is the ordinary simply-connected Euclidean manifold \mathcal{R}^3 . However, as far as multiply-connected spacelike three-spaces \mathcal{M}_3 are concerned the choice of satisfactory boundary conditions for this problem is not as simple as that. So, for example, in the exam of the electric field produced by an isolated point charge in any one of the six possible orientable compact (multiply-connected) topologically distinct Euclidean 3-manifolds \mathcal{M}_3 , clearly this condition at infinity cannot be imposed. Actually, it is easy to show by assuming the Gauss's law that one cannot have a net electric charge in these compact flat manifolds, making clear that the overall electric charge in these manifolds is related to the topology [4]. This example evinces that changes in the assumption of the space-time topology may induce important physical consequences even in the case of static flat space-time manifolds. This type of sensitivity, which is also present in cosmological modelling [17], has been referred to as topological fragility and can clearly occur without violation of any local physical law [1,17]. As a matter of fact, in the above example the topological fragility arises when we impose the local validity of an ordinary law of classical electromagnetism to a system which lies in a flat multiply-connected manifold.

In the same context, in this work we have studied the role played by multiply-connectedness in the time evolution of the energy $E(t)$ of a radiating system that lies in static flat space-time manifolds \mathcal{M}_4 whose $t = \text{const}$ spacelike sections \mathcal{M}_3 are compact in at least one spatial direction. So, it may even have the lowest degree of multiply-connectedness. This topological setting is general enough to encompass the entire set of possible classes of multiply-connected flat 3-manifolds discussed, for example, by Wolf [2]. We have shown that the behavior of the radiating energy $E(t)$ of the oscillator changes remarkably from exponential damping, when the underlying 3-manifold is \mathcal{R}^3 , to a reverberation behavior for $E(t)$ when the spacelike $t = \text{const}$ sections of \mathcal{M}_4 are any possible multiply-connected flat 3-manifold. Thus, our study shows that the topological fragilities can arise not only in the usual cosmological modelling [17], but also in ordinary static flat space-time manifolds as long as they are multiply connected. As we have emphasized in section II and III this striking behavior of $E(t)$ occurs with no violation of any physical law, and has a purely topological origin.

Although the physical system we have investigated here is the same system discussed in [1], the present article generalizes the results of that paper in several respects. Firstly, the underlying topological setting is much more general since we did not assume that our spacelike 3-manifolds \mathcal{M}_3 are compact and orientable. Actually we only require that they are multiply-connected, which means that \mathcal{M}_3 can be endowed with any one of the 17 multiply-connected flat topologies [2], including, of course, the orientable compact 3-manifolds dealt with in [1]. Secondly, contrarily to the *numerical* integration performed in [1], here we have obtained a closed *exact* solution of the evolution equation for the harmonic oscillator in the above-mentioned general topological setting. Thirdly, in the case study we discussed in details in section IV we have found the (exact) time evolution of our system in a specific *non-compact* (although multiply-connected) manifold, while in [1] the time behavior of our system was (numerically) treated only for six *compact orientable* background manifolds. Finally in the present article we have also discussed the conservation and the balance of the total energy of our physical system, making apparent that in both papers the total energy is finite and conserved for any time $t > 0$.

Before closing it is worth mentioning that to the extent that we have studied the dynamical behavior of a test system in *static* flat FRW space-time backgrounds our results have a moderate, though suggestive and well-founded, relation to cosmology. Besides, it should be noticed that the net role played by topology is better singled out in this static case, where the dynamical degrees of freedom have been frozen. On the other hand, it appears that, apart from

the inflationary expansion and perhaps a few other cases, the restriction we have made will not be decisive for the pattern of the behavior radiating energy $E(t)$ in a number of flat expanding locally homogeneous and isotropic FRW cosmological models. Finally, it is worth noting that since geometry constrains, but does not dictate the topology of space-time manifolds, in cosmological modelling one often is confronted with the basic question of what topologies are physically acceptable for a given space-time manifold. An approach to this problem is to study the possible physical (observational) consequences of adopting particular topologies for the space-time (for a fair list of references on this approach see the review article by Lachièze-Rey and Luminet [18]). This paper constitutes an example of this relevant approach.

ACKNOWLEDGEMENTS

We would like to express our thanks to Professor John A. Wheeler for stimulating correspondence on this subject matter. We are also grateful to the Brazilian scientific agencies CAPES, FAPERJ and CNPq for the grants under which this work was carried out.

APPENDIX: POINTLIKE COUPLING LIMIT CASE

Our aim in this appendix is to show how one can derive from equation (2.12), which holds for an arbitrary density function $\rho_n(\vec{x})$ of a δ -sequence, the radiation reaction equation (2.13) for the limiting case of pointlike coupling between the scalar field and the oscillator $[\rho(\vec{x}) \rightarrow \delta^3(\vec{x})]$.

Let us first rewrite the equation (2.12) in the following form

$$\ddot{Q}_n(t) + 2\Gamma I_n(t) + \Omega^2 Q_n(t) = 2\Gamma \sum_{\vec{r} \in \tilde{\mathcal{O}}_q} J_{n,\vec{r}}(t), \quad (\text{A.1})$$

where we have set

$$I_n(t) = \int d^3x d^3y \rho_n(\vec{x}) \rho_n(\vec{y}) \frac{Q_n(t) - \theta(\tau) Q_n(\tau)}{|\vec{x} - \vec{y}|}, \quad (\text{A.2})$$

$$J_{n,\vec{r}}(t) = \int d^3x d^3y \frac{\rho_n(\vec{x}) \rho_{n,\vec{r}}(\vec{y})}{|\vec{x} - \vec{y}|} \theta(\tau) Q_n(\tau), \quad (\text{A.3})$$

with $\tau = t - |\vec{x} - \vec{y}|$, $2\Gamma = \lambda^2/4\pi$ and $\tilde{\mathcal{O}}_q = \mathcal{O}_q - \{(0,0,0)\}$.

As we mentioned earlier the pointlike limit case can be achieved by taking the limit $n \rightarrow \infty$ of the above n -system equation. To this end we firstly remind that as we are dealing with density functions $\rho_n(\vec{x})$ whose supports are connected sets, the following reworded version of the mean value theorem can be stated: let $f(\vec{x}, \vec{y})$ be an arbitrary continuous function defined in $\mathcal{I} = \text{supp}(\rho_n) \times \text{supp}(\rho_{n,\vec{r}})$, then there exists a point $(\vec{x}_0, \vec{y}_0) \in \mathcal{I}$ such that

$$f(\vec{x}_0, \vec{y}_0) = \int d^3x d^3y \rho_n(\vec{x}) \rho_{n,\vec{r}}(\vec{y}) f(\vec{x}, \vec{y}) \quad (\text{A.4})$$

holds.

Let now R_n be the radius of $\text{supp}(\rho_n)$ and consider a point $\vec{r} \in \tilde{\mathcal{O}}_q$, then by the mean value theorem there exists a sequence of pair of points $(\vec{x}_{0,n}, \vec{y}_{0,n}) \in \mathcal{I}$ such that for each $t > 0$ we have the sequence of numbers

$$J_{n,\vec{r}}(t) = \begin{cases} 0 & \text{if } t < |\vec{r}| - 2R_n, \\ \text{unknown} & \text{if } |\vec{r}| - 2R_n < t < |\vec{r}| + 2R_n, \\ Q_n(t - \epsilon_n)/\epsilon_n & \text{if } |\vec{r}| + 2R_n < t, \end{cases} \quad (\text{A.5})$$

where $\epsilon_n = |\vec{x}_{0,n} - \vec{y}_{0,n}|$. Since $R_n \rightarrow 0$ when $n \rightarrow \infty$, the mid subinterval shrinks to the point $t = |\vec{r}|$, then it is irrelevant for our purpose the knowledge of $J_{n,\vec{r}}(t)$ in that region. Furthermore, since $\epsilon_n \rightarrow |\vec{r}|$ we have

$$\lim_{n \rightarrow \infty} J_{n,\vec{r}}(t) = \frac{\theta(t - |\vec{r}|) Q(t - |\vec{r}|)}{|\vec{r}|} . \quad (\text{A.6})$$

The limit of $I_n(t)$ can be obtained as follows. For a given fixed value $t > 0$, there exists a natural number n_0 such that for all $n > n_0$ we have $t > 2R_n$. Thus by the mean value theorem there also exists a pair of points $\vec{x}_{0,n}, \vec{y}_{0,n} \in \text{supp}(\rho_n)$ such that

$$I_n(t) = \frac{Q_n(t) - Q_n(t - \epsilon_n)}{\epsilon_n} , \quad (\text{A.7})$$

with $\epsilon_n = |\vec{x}_{0,n} - \vec{y}_{0,n}| \rightarrow 0$ when $n \rightarrow \infty$. Since $Q_n(t)$ has continuous first derivative we can expand it about the fixed value t up to first order to obtain

$$I_n(t) = \dot{Q}_n(t) + O(\epsilon_n) . \quad (\text{A.8})$$

Therefore the following limit:

$$\lim_{n \rightarrow \infty} I_n(t) = \dot{Q}(t) \quad (\text{A.9})$$

holds for all $t > 0$.

Finally, using equations (A.1), (A.6) and (A.9) one obtains the radiation reaction equation (2.13) for the pointlike limit case $\rho_n(\vec{x}) \rightarrow \delta^3(\vec{x})$.

* E-MAIL ADDRESS: german@cbpf.br

§ E-MAIL ADDRESS: reboucas@cbpf.br

† E-MAIL ADDRESS: teixeira@cat.cbpf.br

‡ On leave of absence from Facultad de Ciencias, Universidad Nacional de Ingeniería, Apartado 31 - 139, Lima 31 - Peru.
E-MAIL ADDRESS: bernui@fc-uni.edu.pe

- [1] A. Bernui, G.I. Gomero, M.J. Rebouças & A.F.F. Teixeira, *Phys. Rev. D* **15** **57**, 4699 (1998).
- [2] J.A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, New York (1967).
- [3] G.F.R. Ellis, *Gen. Rel. Grav.* **2**, 7 (1971).
- [4] G.F.R. Ellis & G. Schreiber, *Phys. Lett. A* **115**, 97 (1986).
- [5] G.I. Gomero, *Fundamental Polyhedron and Glueing Data for the Sixth Euclidean Compact Orientable 3-manifold*, CBPF-NF-049/97, Centro Brasileiro de Pesquisas Físicas report (1997).
- [6] F. Schwalb & W. Thirring, *Ergeb. exakten Naturwiss.* **36**, 219 (1964).
- [7] W. Burke, *J. Math. Phys.* **12**, 401 (1971).
- [8] P. Aichelburg & R. Beig, *Ann. Phys. (N.Y.)* **98**, 264 (1976).
- [9] P. Aichelburg & R. Beig, *Phys. Rev. D* **15**, 389 (1977).

- [10] R. Beig, *J. Math. Phys.* **19**, 1104 (1978).
- [11] W. Unruh, in *Gravitational Radiation*, Les Houches 1982, eds. N. DeRuelle & T. Piran, North-Holland, Amsterdam (1983).
- [12] J. Stewart, *Gen. Rel. Grav.* **15**, 425 (1983).
- [13] J.L. Anderson, *Gen. Rel. Grav.* **16**, 595 (1984).
- [14] C. Hoenselaers & B. Schmidt, *Class. Quantum Grav.* **6**, 867 (1989).
- [15] A. Bernui, *Appl. Anal.* **42**, 157 (1991).
- [16] A. Bernui, *Ann. Physik* **3**, 408 (1994).
- [17] M.J. Rebouças, R.K. Tavakol & A.F.F. Teixeira, *Gen. Rel. Grav.* **30**, 535 (1998).
- [18] M. Lachièze-Rey & J.-P. Luminet, *Phys. Rep.* **254**, 135 (1995). See also references therein.
- [19] N.G. van Kampen, *Dan. Mat. Fys. Medd.* **26**, 16 (1951).
- [20] H.A. Kramers, *Collected Scientific Papers*, p. 845, North-Holland, Amsterdam (1956).
- [21] B.F. Schutz, in *Relativistic Astrophysics and Cosmology*, p. 35–97, eds. X. Fustero & E. Verdaguer, World Scientific, Singapore (1984).
- [22] E. Balbiniski, S.L. Detweiler & B.F. Schutz *Mon. Not. Roy. Astron. Soc.* **213**, 553 (1985).
- [23] K.D. Kokkotas & B.F. Schutz, *Gen. Rel. Grav.* **18**, 913 (1986).
- [24] R.D. Richtmyer, *Principles of Advanced Mathematical Physics. Vol.I*, Springer-Verlag, New York (1978).
- [25] K.M. Heal, M.L. Hansen & K.M. Rickard, *Maple V Learning Guide*, Springer-Verlag, New York (1996).
- [26] A. Heck, *Introduction to Maple*, Springer-Verlag, New York (1993).

FIG. 1. Two-dimensional schematic figure of a pointlike harmonic oscillator at a point $q \in \mathcal{M}_3$ and its equivalent system on the covering manifold \mathcal{R}^3 , which is formed by an infinite set of identical oscillators each one located at a point of the discrete orbit $\pi^{-1}(q) = \mathcal{O}_q \subset \mathcal{R}^3$. All oscillators interact identically with the scalar field φ and are subject to the same set of initial conditions.

FIG. 2. Schematic representation of the curves corresponding to the left- and right-hand side of equation (3.22), whose intersection in just one point evinces that this equation has a single real positive solution $\sigma = b$.

FIG. 3. Graph of the amplitude $Q(t)$ for the overdamped harmonic oscillator with $\Gamma = 6$ and $\Omega = 5$ in a flat space-time whose $t = \text{const}$ spacelike section \mathcal{M}_3 is endowed with the topology $\mathcal{R}^2 \times S^1$, and where we have taken $a = 1$ in eq. (4.1), and as initial conditions $Q(0) = 1$ and $\dot{Q}(0) = 0$. There are a few relative maxima and minima followed by a growth of $Q(t)$. This time evolution sharply contrasts with the behavior of the amplitude $Q(t)$ for this overdamped oscillator when $\mathcal{M}_3 = \mathcal{R}^3$, where there are neither relative extrema nor growth of $Q(t)$. This figure also contains the graph of the function $\exp(-bt)Q(t)$, which within the accuracy of the plot confirms an exponential form for the divergent behavior suggested by the graph of $Q(t)$. To have this last graph we have used $b = 0.35$, which is the approximate root of eq. (4.2).

FIG. 4. Behavior of the energy $E(t)$ for the overdamped harmonic oscillator with $\Gamma = 6$ and $\Omega = 5$ in a flat space-time whose $t = \text{const}$ spacelike sections \mathcal{M}_3 have the topology $\mathcal{R}^2 \times S^1$, and where we have taken $a = 1$ in eq. (4.1), and as initial conditions $Q(0) = 1$ and $\dot{Q}(0) = 0$. The graph for $E(t)$ exhibits a few relative minima and maxima followed by an asymptotic divergent behavior. This divergent conduct is clearly conform to the behavior of $Q(t)$ when $t \rightarrow \infty$. The time evolution of the energy for the overdamped harmonic oscillator in this multiply-connected manifold is in striking contrast with the behavior of the energy $E(t)$ in the simply-connected case \mathcal{R}^3 , where there is an exponential damping of $E(t)$ with no relative extrema. This figure also contains the graph of the derivative $\dot{E}(t)$, exhibiting finite discontinuities at each $t = t_k = k a$ (k stands for a positive integer), which again do not occur in the simply-connected case \mathcal{R}^3 . The graph of $\dot{E}(t)$ also reveals that $E(t)$ has a finite number of extrema. The relative minima and maxima of the energy $E(t)$, the discontinuities of the derivative $\dot{E}(t)$, and the asymptotic divergent behavior, which take place in multiply-connected flat manifolds, all have a topological origin.







